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Calculation of TM_{0n} Dispersion Relations in a Corrugated Cylindrical Waveguide

ALAN BROMBORSKY, MEMBER, IEEE, AND BRIAN RUTH

***Abstract*—The TM_{0n}-mode Maxwell equations in a cylindrical geometry are converted to a state-vector system of coupled linear differential equations, in which the boundary conditions for a waveguide of varying diameter are included in the coefficient matrix of the state-vector system. The particular problem of periodic boundary conditions is solved for a waveguide with a sinusoidally undulating wall.**

I. Introduction

THE GENERATION of ultra-high-power (~1 GW) microwave pulses, via the driving of slow-wave structures by intense, pulsed, relativistic electron beams (0.5 to 2.0 MeV, 2 to 15 kA, 15 to 100 ns) [1], [2], places unique demands upon the slow-wave structure in terms of the RF power densities (0.3 GW/cm^2) and electric fields (400 kV/cm) present in the structure. Conventional slow-wave structures, such as the helix- and iris-loaded waveguides, are susceptible to high-field breakdown, and hence plasma formation, with the subsequent shorting out of the slow-wave structure. What is required for ultra-high-power devices is a structure with a periodic wall shape that does not lead to undue electric-field intensification. A possible candidate is a cylindrical guide in which the waveguide diameter varies sinusoidally with axial position.

However, in order to design a device utilizing such a structure, the cold waveguide dispersion relation and the electromagnetic field distribution must be accurately determined.

The basic objective of this paper is to describe a technique for computing the dispersion relation and electromagnetic fields of the TM_{0n} modes of a periodically rippled cylindrical waveguide. Please note that the technique to be described also can be applied to other than TM_{0n} modes, so that with minor changes the derivation could be quite useful in calculating TE modes in tapered gyrotron cavities. Also note that the source terms in the Maxwell equations are not initially set to zero. This is done so that eventually the field calculation described can be used to compute the coupling impedance between an electron beam and a propagating waveguide mode.

II. Scaling of Maxwell Equations

A. Notation

We define (in MKS units)

c	free-space speed of light,
ϵ_0	permittivity of free space,
μ_0	permeability of free space,
η_0	free-space wave impedance (377 Ω),
ω	wave circular frequency,
r, θ, z	cylindrical coordinates,
E_r, E_θ, E_z	electric-field components,
H_r, H_θ, E_z	magnetic-field components,
J_r, J_θ, J_z	current-density components,
L	periodicity length of slow-wave structure,

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J_0	scaling current density,
σ	conductivity of waveguide wall,
$R(z)$	radius of slow-wave structure as a function of axial position,
i	square root of minus one,
$(\,)'$	differentiation operator,
$(\,)^T$	transpose operator,
$(\,)^*$	conjugation operator,
$(\,)^\dagger$	hermitian conjugation operator.

B. Transformation of Equations

The oscillatory Maxwell equations, with an assumed time dependence of $e^{i\omega t}$, for the azimuthally symmetric TM modes (E_θ, H_r, H_z, and J_θ all zero) are

$$-\frac{\partial H_\theta}{\partial z} = i\omega\epsilon_0 E_r + J_r \tag{1a}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rH_\theta) = i\omega\epsilon_0 E_z + J_z \tag{1b}$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -i\omega\mu_0 H_\theta . \tag{1c}$$

For computational purposes, and in order to understand how the dispersion relations scale with system dimensions, the Maxwell equations should be rewritten in terms of dimensionless variables. Introducing the variables

$$\Omega = \omega L/c \tag{2a}$$

$$\xi = z/L \tag{2b}$$

$$\mu = r/L \tag{2c}$$

$$S_r = J_r/J_0 \tag{2d}$$

$$S_z = J_z/J_0 \tag{2e}$$

$$F_\theta = H_\theta/(LJ_0) \tag{2f}$$

$$G_r = i\Omega E_r/(\eta_0 L J_0) \tag{2g}$$

$$G_z = i\Omega E_z/(\eta_0 L J_0) \tag{2h}$$

and substituting them into the first equation set (1) transforms the Maxwell equations into

$$\frac{\partial F_\theta}{\partial \xi} = -(G_r + S_r) \tag{3a}$$

$$\frac{1}{\mu}\frac{\partial}{\partial \mu}(\mu F_\theta) = G_z + S_z \tag{3b}$$

$$\frac{\partial G_r}{\partial \xi} - \frac{\partial G_z}{\partial \mu} = \Omega^2 F_\theta . \tag{3c}$$

Note that, in addition to making the Maxwell equations dimensionless, the change of variables has eliminated the frequency coefficients from all equations except (3c). The frequency coefficient Ω^2 in (3c) is now real if Ω is real.

III. Waveguide Boundary Conditions

If the waveguide is a perfect conductor, the boundary condition to be enforced is zero tangential electric field on the boundary, $E_t(R(z), z) = 0$. If the waveguide wall is not a perfect conductor, the tangential electric field is related to the azimuthal magnetic field by

$$E_t(R(z), z) = -Z_w H_\theta(R(z), z) \tag{4}$$

where the surface impedance of the waveguide wall Z_w is [3]

$$Z_w = (1+i)\sqrt{(\Omega\eta_0)/(2\sigma L)} . \tag{5}$$

In order to express (4) in terms of the same dimensionless variables as the system (3), define a dimensionless waveguide radius by

$$\alpha(\xi) = R(L\xi)/L \tag{6}$$

and substitute (2f), (2g), and (2h) into (4) to obtain

$$G_t(\alpha(\xi), \xi) = -\left[i(1+i)\Omega^{3/2}/\sqrt{2\sigma L\eta_0}\right] F_\theta(\alpha(\xi), \xi). \tag{7}$$

Defining

$$\eta_w(\Omega) = -i(1+i)\Omega^{3/2}/\sqrt{2\sigma L\eta_0} \tag{8}$$

and expressing $G_t(\alpha(\xi), \xi)$ in terms of its components produces a boundary equation of the form

$$\left(G_z(\alpha(\xi), \xi) + \alpha'(\xi) G_r(\alpha(\xi), \xi)\right)/\sqrt{1+\alpha'(\xi)^2} = \eta_w(\Omega) F_\theta(\alpha(\xi), \xi) \tag{9}$$

which allows one to express G_z on the wall in terms of G_r and F_θ

$$G_z(\alpha(\xi), \xi) = \sqrt{1+\alpha'(\xi)^2}\, \eta_w(\Omega) F_\theta(\alpha(\xi), \xi) - \alpha'(\xi) G_r(\alpha(\xi), \xi). \tag{10}$$

In addition to the boundary conditions at the waveguide wall, the cylindrical symmetry of the system requires that, on the waveguide axis ($\mu = 0$), F_θ, G_r, and S_r be zero for all ξ.

IV. Transformation of Maxwell Equations to State Vector Equations

A. Notation

Denote a general state vector $\vec{V}(\xi)$ by

$$\vec{V}(\xi) = [V_0(\xi), \cdots, V_{N+1}(\xi)]^T \tag{11}$$

and define a basis set vector by

$$\vec{\phi}(\nu) = [\phi_0(\nu), \cdots, \phi_{N+1}(\nu)]^T \tag{12}$$

where the $\phi_j(\nu)$ form a complete set of at least once-differentiable basis functions on the interval $(0,1)$ as $N \to \infty$, and let $\vec{\phi}(0) = \vec{0}$. For the variable-diameter cylindrical waveguide, let

$$\nu(\mu, \xi) = \mu/\alpha(\xi) \tag{13}$$

so that the waveguide radius for any given ξ is equivalent to $\nu = 1$.

B. Field Expansions

Except for $G_z(\mu, \xi)$, which would be redundant, expand all field and source components in terms of the basis set vector and a state vector

$$F_\theta(\mu,\xi)=\vec{\phi}(\nu(\mu,\xi))^T\vec{F}(\xi) \quad (14)$$

$$G_r(\mu,\xi)=\vec{\phi}(\nu(\mu,\xi))^T\vec{G}(\xi) \quad (15)$$

$$S_r(\mu,\xi)=\vec{\phi}(\nu(\mu,\xi))^T\vec{S}^r(\xi) \quad (16)$$

$$S_z(\mu,\xi)=\vec{\phi}(\nu(\mu,\xi))^T\vec{S}^z(\xi)/\nu(\mu,\xi). \quad (17)$$

Because $\vec{\phi}(0)=\vec{0}$, the boundary conditions on F_θ, G_r, and S_r are satisfied at $\mu=0$. However, so that S_z can take on nonzero values at $\mu=0$, the expansion for S_z (17) is modified by the factor $1/\nu$. Then, if $\vec{\phi}(\nu)$ is restricted to go to zero linearly as ν approaches 0, $S_z(0,\xi)$ can take on finite nonzero values. The field expansion for $G_z(\mu,\xi)$ is obtained by substituting (14) and (17) into (3b)

$$G_z(\mu,\xi)=\left(\vec{\phi}'(\nu)+\vec{\phi}(\nu)/\nu\right)^T\vec{F}(\xi)/\alpha(\xi) - \vec{\phi}(\nu)^T\vec{S}^Z(\xi)/\nu. \quad (18)$$

C. State Vector Form of Curl H Equation (3a)

Substituting (14), (15), and (16) into (3a) yields

$$\frac{\partial\nu}{\partial\xi}\vec{\phi}'(\nu)^T\vec{F}+\vec{\phi}(\nu)^T\frac{d\vec{F}}{d\xi}=-\vec{\phi}(\nu)^T(\vec{G}+\vec{S}^r). \quad (19)$$

Simplifying $\partial\nu/\partial\xi$ by noting

$$\frac{\partial\nu}{\partial\xi}=-\frac{\alpha'(\xi)}{\alpha(\xi)}\frac{\mu}{\alpha(\xi)}=-\frac{\alpha'}{\alpha}\nu \quad (20)$$

and substituting (20) into (19), we obtain

$$\vec{\phi}(\nu)^T\frac{d\vec{F}}{d\xi}=\frac{\alpha'}{\alpha}\nu\vec{\phi}'(\nu)^T\vec{F}-\vec{\phi}(\nu)^T(\vec{G}+\vec{S}^r). \quad (21)$$

Multiply (21) from the left by $w(\nu)\vec{\phi}(\nu)$ where $w(\nu)$ is an arbitrary weight function. Then integrate the resulting equation from 0 to 1 with respect to ν, giving

$$A\frac{d\vec{F}}{d\xi}=\frac{\alpha'(\xi)}{\alpha(\xi)}B\vec{F}-A(\vec{G}+\vec{S}^r) \quad (22)$$

where A and B are dyadic matrices defined by

$$A=\int_0^1 d\nu\, w(\nu)\vec{\phi}(\nu)\vec{\phi}(\nu)^T \quad (23)$$

$$B=\int_0^1 d\nu\, w(\nu)\nu\vec{\phi}(\nu)\vec{\phi}'(\nu)^T. \quad (24)$$

D. State Vector Form of Curl E Equation (3c)

Substitute (14) and (15) into (3c)

$$\vec{\phi}(\nu)^T\frac{d\vec{G}}{d\xi}=\frac{\partial G_z}{\partial\mu}+\frac{\alpha'}{\alpha}\nu\vec{\phi}'(\nu)^T\vec{G}+\Omega^2\vec{\phi}(\nu)^T\vec{F}. \quad (25)$$

Multiply (25) from the left by $w(\nu)\vec{\phi}(\nu)$ and integrate from 0 to 1 with respect to ν, giving

$$A\frac{d\vec{G}}{d\xi}=\int_0^1 d\nu\, w(\nu)\vec{\phi}(\nu)\frac{\partial G_z}{\partial\mu}+\frac{\alpha'(\xi)}{\alpha(\xi)}B\vec{G}+\Omega^2A\vec{F}. \quad (26)$$

Now reduce the first term of the right-hand side (r.h.s.) of (26) via integration by parts

$$\int_0^1 d\nu\, w(\nu)\vec{\phi}(\nu)\frac{\partial G_z}{\partial\mu}=\frac{1}{\alpha(\xi)}\int_0^1 d\nu\, w(\nu)\vec{\phi}(\nu)\frac{\partial G_z}{\partial\nu} = \frac{1}{\alpha}\left\{\left[w(\nu)\vec{\phi}(\nu)G_z\right]_0^1-\int_0^1 d\nu\left(w(\nu)\vec{\phi}(\nu)\right)'G_z\right\}. \quad (27)$$

Evaluate the first term in the r.h.s. of (27) by using the boundary condition (10) for which the expansions (14) and (15) for $F_\theta(\alpha(\xi),\xi)$ and $G_r(\alpha(\xi),\xi)$ have been substituted into it

$$\frac{1}{\alpha}\left[w(\nu)\vec{\phi}(\nu)G_z\right]_0^1 = \frac{\sqrt{1+(\alpha')^2}}{\alpha}\eta_w(\Omega)w(1)\vec{\phi}(1)\vec{\phi}(1)^T\vec{F}(\xi) - \frac{\alpha'}{\alpha}w(1)\vec{\phi}(1)\vec{\phi}(1)^T\vec{G}(\xi). \quad (28)$$

Evaluate the second term in the r.h.s. of (27) by substituting (18)

$$\frac{1}{\alpha}\int_0^1 d\nu\left(w(\nu)\vec{\phi}(\nu)\right)'G_z=\frac{1}{\alpha^2}(C+D)\vec{F}(\xi)-\frac{1}{\alpha}D\vec{S}^z(\xi) \quad (29)$$

where C and D are the dyadic matrices

$$C=\int_0^1 d\nu\left(w(\nu)\vec{\phi}(\nu)\right)'\vec{\phi}'(\nu)^T \quad (30)$$

$$D=\int_0^1 d\nu\left(w(\nu)\vec{\phi}(\nu)\right)'\vec{\phi}(\nu)^T/\nu. \quad (31)$$

Now substitute (28) and (29) into (26) via (27) to obtain the Curl E state vector equation

$$A\frac{d\vec{G}}{d\xi}=\frac{\alpha'(\xi)}{\alpha(\xi)}\left(B-w(1)\vec{\phi}(1)\vec{\phi}(1)^T\right)\vec{G} + \left\{\frac{\sqrt{1+\alpha'(\xi)^2}}{\alpha(\xi)}\eta_w(\Omega)w(1)\vec{\phi}(1)\vec{\phi}(1)^T + \Omega^2A-\frac{1}{\alpha(\xi)^2}(C+D)\right\}\vec{F} + \frac{1}{\alpha(\xi)}D\vec{S}^z(\xi). \quad (32)$$

E. Total State Vector Equations

Define

$$\vec{Q}(\xi)^T=\left(\vec{F}(\xi)^T,\vec{G}(\xi)^T\right) \quad (33)$$

$$\vec{S}(\xi)^T=\left(\vec{S}^r(\xi)^T,\vec{S}^z(\xi)^T\right) \quad (34)$$

$$I=\begin{bmatrix}A & 0\\ 0 & A\end{bmatrix} \quad (35)$$

$$M(\xi)=\begin{bmatrix} \frac{\alpha'}{\alpha}B & -A \\ \sqrt{\frac{1+(\alpha)^2}{\alpha}}\eta_w(\Omega)w(1)\vec{\phi}(1)\vec{\phi}(1)^T+\Omega^2 A-\frac{(C+D)}{\alpha^2} & \frac{\alpha'}{\alpha}\left(B-w(1)\vec{\phi}(1)\vec{\phi}(1)^T\right)\end{bmatrix} \quad (36)$$

$$N(\xi)=\begin{bmatrix} -A & 0 \\ 0 & D/\alpha \end{bmatrix} \quad (37)$$

so that the state vector equation can be written

$$I\frac{d\vec{Q}}{d\xi}=M(\xi)\vec{Q}+N(\xi)\vec{S}. \quad (38)$$

V. Solution of State Vector Equations for Periodic Boundary Condition

A. Floquet Solution

If $R(z)$ is periodic with period L then $\alpha(\xi)$ is periodic with period 1 and likewise α'/α, $\sqrt{1+(\alpha')^2}/\alpha$, and $1/\alpha^2$ are also periodic with period 1, implying that $M(\xi)$ and $N(\xi)$ are also periodic with period one. Assuming now, and that for the remainder of this paper, that the source term $\vec{S}(\xi)$ is identically zero, Floquet's theorem [4] states that there exists a complete set of solutions to (38) of the form

$$\vec{Q}(\xi)=e^{\Gamma\xi}\vec{q}(\xi) \quad (39)$$

where

$$\vec{q}(\xi)=\vec{q}(\xi+1). \quad (40)$$

The dispersion relation of the empty waveguide is the implicit dependence of pure imaginary Γ's upon the parameter Ω. The differential equation of a Floquet solution is obtained by substituting (39) into (38)

$$I\frac{d\vec{q}}{d\xi}=(M(\xi)-\Gamma I)\vec{q}. \quad (41)$$

B. Fourier Series Solution of Floquet Equation (41)

Expand both $M(\xi)$ and $\vec{q}(\xi)$ in a complex Fourier series

$$M(\xi)=\sum_{j=-\infty}^{\infty} M_j e^{2\pi ij\xi} \quad (42)$$

$$\vec{q}(\xi)=\sum_{j=-\infty}^{\infty} \vec{q}_j e^{2\pi ij\xi} \quad (43)$$

and substitute (42) and (43) into (41)

$$\sum_{j=-\infty}^{\infty} 2\pi ijI\vec{q}_j e^{2\pi ij\xi}=\sum_{j=-\infty}^{\infty}\left\{\left(\sum_{l=-\infty}^{\infty} M_{j-l}\vec{q}_l\right)-\Gamma I\vec{q}_j\right\}e^{2\pi ij\xi}. \quad (44)$$

Equating like powers of $e^{2\pi i\xi}$ in (44) yields

$$\sum_{l=-\infty}^{\infty} M_{j-l}\vec{q}_l-(\Gamma+2\pi ij)I\vec{q}_j=\vec{0} \quad (45)$$

which reduces the solution of Γ and $\vec{q}$ in (41) to a generalized matrix eigenvalue problem of the form $(W-\Gamma U)\,\vec{x}=\vec{0}$, where the elements of W and U are constructed from the matrices M_k and I as defined by (45). If $M(\xi)$ is real ($\eta_w=0$), there are equivalent real forms of W and U which would simplify and speed up the numerical computation of the eigenvalues of the system.

VI. Selection of Basis Vector and Weight Function

A critical point in setting up the systems of equations (45) to be solved is the choice of the basis set vector (12). The $\phi_j(\nu)$ should be simple in form to allow for easy calculation of the matrices A, B, C, and D, and yet be of a form that relatively few (N small) are required to well approximate the electromagnetic fields in the field expansions. It is tempting to let $\phi_j(\nu)=J_0'(p_{0,j+1}\nu)$, the derivative of the zero-order Bessel function where $p_{0,n}$ is the nth zero of $J_0(x)$, since such a choice for $\phi_j(\nu)$ would allow the state vector equations to satisfy exactly the constant diameter waveguide case ($\alpha(\xi)=$ constant) for a finite N. The problem with this choice is that $\vec{\phi}(1)=\vec{0}$, but if one allows the walls of the guide to undulate $G_z(\alpha(\xi),\xi)\neq 0$. Such a choice of basis vector does not allow the axial electric-field expansion to converge on the waveguide boundary, and leads to Gibbs' oscillations near the boundaries. At the suggestion of Dr. David Russel (U.S. Army Mathematics Center, University of Wisconsin, Madison) the piecewise cubic cardinal spline functions [5] were chosen for the $\phi_j(\nu)$. However, since not all splines are zero for $\nu=0$, the splines were modified in the following manner. Denoting the cardinal splines on the unit interval subdivided into N subintervals by $B_j(N;\nu)$, one has the problem that $B_0(N;0)$ and $B_1(N;0)$ are not zero. This problem is rectified by defining

$$\phi_0(\nu)=B_0(N;\nu)-4B_{-1}(N;\nu) \quad (46)$$

$$\phi_1(\nu)=B_1(N;\nu)-B_{-1}(N;\nu) \quad (47)$$

$$\phi_j(\nu)=B_j(N;\nu), \qquad 1<j\leqslant N+1 \quad (48)$$

where

$$B_0(N;\nu)=\begin{cases} 0, & \nu\leqslant -2/N \\ (N\nu+2)^3, & -2/N<\nu<-1/N \\ 1+3(N\nu+1)+3(N\nu+1)^2-3(N\nu+1)^3, & -1/N\leqslant\nu\leqslant 0 \\ 1-3(N\nu-1)+3(N\nu-1)^2-3(N\nu-1)^3, & 0\leqslant\nu\leqslant 1/N \\ -(N\nu-2)^3, & 1/N\leqslant\nu\leqslant 2/N \\ 0, & \nu\geqslant 2/N \end{cases} \quad (49)$$

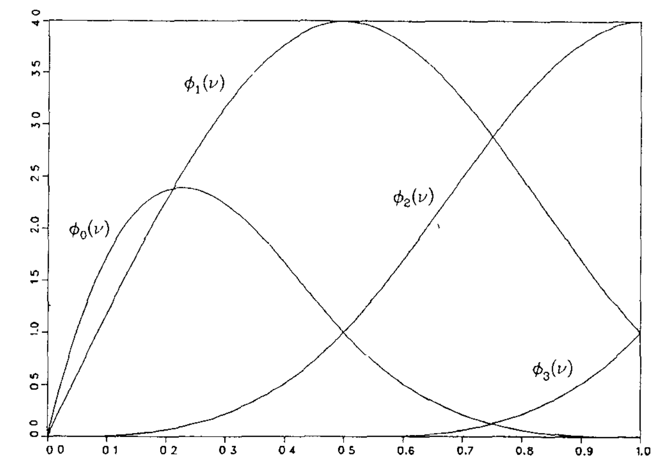

Fig. 1. Component functions of basis vector for $N = 2$.

and

$$B_j(N;\nu) = B_0(N;\nu - j/N). \tag{50}$$

For $N = 2$, the basis functions $\phi_j(\nu)$ are plotted in Fig. 1.

The weight function $w(\nu)$ is now chosen

$$w(\nu) = \nu \tag{51}$$

so that products of the form $\vec{F}^{\dagger}(\xi)A\vec{G}(\xi)$ and other similar products are equivalent to surface integrals of the empty waveguide fields over the cross section of the waveguide. Such a choice of $w(\nu)$ will allow one to relate the coupling impedances of driven modes to physically meaningful quantities such as the axial power flow and the axial electric field of the empty waveguide modes. The derivation of the coupling impedance will be detailed in future publications.

VII. Testing of Basis Vector Sets

To check the choice of basis functions, consider (45) for $\alpha(\xi) = 1$ (no undulation) and $\Omega = 0$

$$(M_0 - \Gamma I)\vec{q}_0 = \vec{0}. \tag{52}$$

From the analytical theory of cylindrical waveguides, one knows that the Γ's satisfying (52) should be the roots, p_{0j}, of the zero-order Bessel function. A comparison of p_{0j} and the Γ's of (52) for various values of N is given in Table I.

As is evident from Table I, $N = 2$ offers accurate values for both the TM_{01} and TM_{02} modes of a constant diameter cylindrical wave guide, and suggests that a basis set vector with four dimensions should suffice when $\alpha(\xi)$ is not constant. To test the technique for a periodic waveguide, the form of $\alpha(\xi)$ was assumed to be

$$\alpha(\xi) = \alpha_0(1 + \delta\cos(2\pi\xi)). \tag{53}$$

The system matrix (36) was expanded into a Fourier series using an FFT program and the M_{j-l} in (45) evaluated for $|j - l| \leq N_{\text{harm}}$. Thus, it was assumed that $M(\xi)$ and $\vec{q}(\xi)$ could be represented adequately with a Fourier series containing N_{harm} harmonics. The eigenvalue problem presented by the truncated form of (45) was solved with the EISPACK subroutine library [6]. The system solved to test the method set $\alpha_0 = 1$ and $\delta = 0.1$. First the system was expanded with $N = 2$ and $N_{\text{harm}} = 1$, and the resulting dispersion curve was plotted in Fig. 2. This plot shows the typical passband stopband behavior of a periodic waveguide system. In order to determine the accuracy of the expansion used, the upper cutoff frequency of the lower branch of the TM_{01} mode ($\text{Im}(\Gamma) = \pi$) was calculated as a function of N and N_{harm}. The results are shown in Table II. The value for $N = 2$, $N_{\text{harm}} = 1$ is within ± 0.11 percent of the value for $N = 4$, $N_{\text{harm}} = 3$ indicating that, at least for the lower branch of the TM_{01} mode, the expansion with $N = 2$, $N_{\text{harm}} = 1$ is accurate enough for all practical applications. If δ is increased, the test described should be performed again to see if $N_{\text{harm}} = 1$ is sufficient. Likewise, the convergence of higher order mode cutoff frequencies

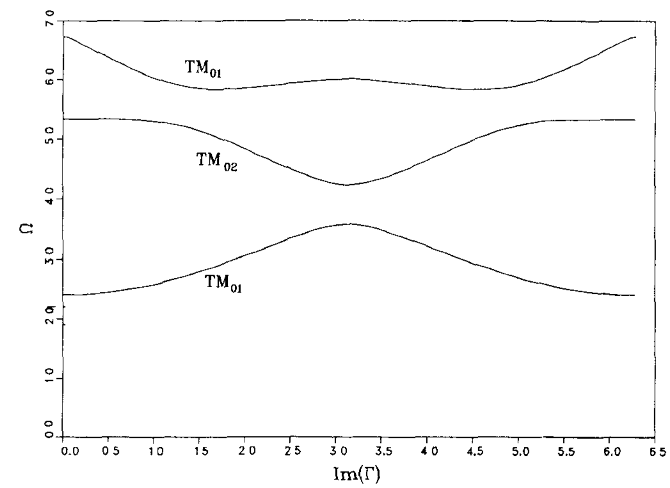

Fig. 2. Dispersion curves for slow-wave structure with $\alpha_0 = 1$ and $\delta = 0.1$.

TABLE I
Comparison of the Roots (p_{0j}) of $J_0(x)$ with Eigenvalues Γ of ($M_0 - \Gamma I$) for Different Values of N

p_{0j} / N	2.4048	5.5200	8.6537	11.791	14.930	18.071
2	2.4049	5.5375	8.8905	17.471	-	-
3	2.4048	5.5227	8.7441	12.418	23.324	-
4	2.4048	5.5205	8.6780	11.991	16.128	29.658
5	2.4048	5.5305	8.6586	11.875	15.235	19.957
6	2.4048	5.5201	8.6550	11.814	15.112	18.462

TABLE II
TM_{01} Cutoff Frequency as a Function of System Expansion Parameters

		N_{harm}		
		1	2	3
N	2	3.584 ± 0.1%	3.577 ± 0.07%	3.574 ± 0.06%
	3	3.581 ± 0.06%	3.576 ± 0.03%	3.576 ± 0.03%
	4	3.581 ± 0.06%	3.573 ± 0.03%	3.576 ± 0.03%

should be checked before complete dispersion curves are computed.

VIII. Results and Conclusions

In the foregoing sections, a technique had been developed to convert the Maxwell equations with boundary conditions to a system of ordinary coupled linear differential equations. The technique is equivalent to that used by S. A. Schelkunoff [7], except that the notation used clarifies the derivations and the basis vector has been chosen so that field convergence is obtained on the waveguide boundary, as well as in the waveguide interior. When applied to a periodic waveguide, the technique allows one to calculate dispersion curves accurately with relatively low-order expansion parameters (N and N_{harm}) and is not limited to fundamental waveguide modes. The technique is also well suited for calculating the coupling impedances required in the normal mode expansion of a driven waveguide (to be described in a future paper).

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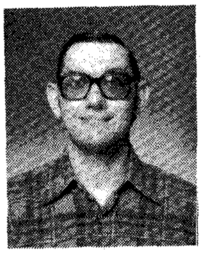

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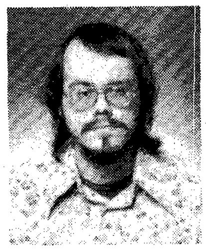

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